

# Dynamical mass generation by source inversion: Calculating the mass gap of the chiral Gross-Neveu model

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We probe the  $U(N)$  chiral Gross-Neveu model with a source term  $J\bar{\Psi}\Psi$ . We find an expression for the renormalization scheme and scale invariant source  $\hat{J}$ , as a function of the generated mass gap. The expansion of this function is organized in such a way that all scheme and scale dependence is reduced to one single parameter  $d$ . We obtain a nonperturbative mass gap as the solution of  $\hat{J}=0$ . A physical choice for  $d$  gives good results for  $N>2$ . The self-consistent minimal sensitivity condition gives a slight improvement.

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## I. INTRODUCTION

In a previous paper, we developed a method for dynamical mass generation in asymptotically free quantum field theories. It was applied to the ordinary Gross-Neveu model [1] and a mass gap was found [2] which agreed very well with the exact result [3]. In this paper we will apply the same method to the non-Abelian Thirring model (NATM) or chiral Gross-Neveu model (CGNM) [1]. It is another one of those rare quantum field theories where exact results, such as the mass gap, can be obtained. In [4] the mass gap is calculated exactly in terms of  $\Lambda$  which is the nonperturbative mass parameter which sets the scale for the running coupling in a certain scheme. Comparing our results with [4] will provide another check on the accuracy of our method.

The idea behind the method is very simple. A source term  $J\bar{\Psi}\Psi$  is added to the NATM Lagrangian and then we calculate the mass gap using ordinary perturbation theory to obtain the perturbative expansion for  $m(J)$ . As a consequence of asymptotic freedom, this expansion is only valid for large values of  $J$ . If we let  $J$  approach zero, the coupling constant grows too large, and the perturbative expansion for  $m(J)$  becomes invalid. Therefore, we cannot take the limit  $J\rightarrow 0$ . If instead we consider the perturbative expansion for the inverted relation  $J(m)$ , perturbation theory remains valid in the limit  $J\rightarrow 0$ , provided that a solution  $m$  exists for  $J(m)=0$ , which is not too small. As in ordinary perturbation theory, the result for the mass gap  $m$  is renormalization scheme (RS) and scale dependent. To eliminate the mass renormalization dependence we use the scheme and scale independent quantity  $\hat{J}$ , instead of  $J$ . Exchanging  $g^2(\mu)$  for  $1/(\beta_0\ln(\mu^2/\Lambda^2))$  as the expansion parameter reduces the remaining dependence to one single number  $d$ , which can be fixed by some external physical condition, or in a more self-consistent approach, by the principle of minimal sensitivity.

The paper is organized as follows. In Sec. II we will present the results necessary for application of the source

inversion for the NATM. In order to avoid unnecessary repetition, we shall refer to the paper [2] for the derivation of the general formula. The outcome of our calculations will be discussed in Sec. III. As a bonus we will show that reparametrization of the  $d$  dependence will enable us to solve the mass gap equation exactly. Details of the exact evaluation of the finite parts of the two loop Feynman integrals which occur in the sunset topology are given in the Appendix.

## II. THE NON-ABELIAN THIRRING MODEL

The  $U(N)$  invariant NATM describes the interaction of  $N$  single flavor Dirac fermions  $\Psi_a, a=1, \dots, N$  in two dimensions with the (massless) Lagrangian

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - \frac{1}{2}g^2(\bar{\Psi}\gamma^\mu T^i\Psi)^2 \quad (1)$$

where  $T^i, i=1, \dots, N^2-1$ , are the generators of  $SU(N)$  with the normalization  $\text{Tr}(T^i T^j) = \frac{1}{2}\delta^{ij}$ . (Note that our coupling constant  $g^2$  is two times the coupling constant  $g^2$  of [4].) This model is also known as the CGNM because a Fierz transformation of the interaction term leads to the equivalent Lagrangian

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi + \frac{g^2}{4}((\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma^5\Psi)^2) + \frac{g^2}{4N}(\bar{\Psi}\gamma^\mu\Psi)^2. \quad (2)$$

The NATM is asymptotically free [5] and possesses, apart from the  $U(N)$  invariance, a chiral  $U(1)$  symmetry. In ordinary perturbation theory this symmetry remains unbroken and no mass gap is generated.

We begin by perturbing Eq. (1) with a  $\bar{\Psi}\Psi$  composite operator to produce the new Lagrangian

$$\mathcal{L}_J = i\bar{\Psi}\not{\partial}\Psi - J\bar{\Psi}\Psi - \frac{g^2}{2}(\bar{\Psi}\gamma^\mu T^i\Psi)^2. \quad (3)$$

The detailed three loop renormalization of this model using dimensional regularization has been given in [6,7] which built on the one and two loop calculations of [5,8,9,6]. The results for the  $\beta$  and  $\gamma$  functions in the modified minimal subtraction (MS) scheme are [ $g^2 = g^2(\mu)$ ]

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} J \Big|_{J_0, g_0, \epsilon} &\equiv -J\gamma(g^2) \equiv -J(\gamma_0 g^2 + \gamma_1 g^4 + \gamma_2 g^6 + \dots) \\ &= -J \left( \frac{(N^2-1)}{2\pi N} g^2 + \frac{(N^2-1)(N-4)}{16\pi^2 N} g^4 + \frac{(N^2-1)(16N^2-12N^3+3N^4+5N^2-26)}{128\pi^3 N^3} g^6 + \dots \right) \end{aligned} \quad (4)$$

and

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} g^2 \Big|_{g_0, \epsilon} &\equiv \beta(g^2) = -2(\beta_0 g^4 + \beta_1 g^6 + \beta_2 g^8 + \dots) \\ &= -2 \left( \frac{N}{4\pi} g^4 - \frac{N}{8\pi^2} g^6 + \frac{5N + \frac{3}{2}N^3 - \frac{11}{2}N + \frac{39}{2}\frac{1}{N}}{64\pi^3} g^8 + \dots \right). \end{aligned} \quad (5)$$

To apply the source inversion at two loop order we also require the two loop perturbative result for the mass gap. To determine this we have computed the fermion two point function at two loops and extracted the finite part exactly after performing the renormalization. The values for the integrals we obtained have been checked against the numerical results of [10] for the mass gap of the ordinary Gross Neveu model. We find

$$\begin{aligned} m(J) = J &\left[ 1 - \frac{(N^2-1)}{N} \left( \ln \frac{J^2}{\mu^2} + 1 \right) \frac{g^2}{4\pi} + \frac{(N^2-1)}{2N} \left( 2\zeta(2) \left( 3 - \frac{1}{N} \right) \right. \right. \\ &\left. \left. + \frac{(3N^2-2N-4)}{N} + \frac{(7N^2+4N-6)}{N} \ln \frac{J^2}{\mu^2} + \frac{(2N^2-1)}{N} \ln^2 \frac{J^2}{\mu^2} \right) \frac{g^4}{16\pi^2} + \dots \right] \end{aligned} \quad (6)$$

where  $\zeta(n)$  is the Riemann zeta function. From this one easily arrives at the expansion for the inverted relation

$$\begin{aligned} J(m) = m &\left[ 1 + \frac{(N^2-1)}{N} \left( \ln \frac{m^2}{\mu^2} + 1 \right) \frac{g^2}{4\pi} + \frac{(N^2-1)}{2N} \left( 2\zeta(2) \left( \frac{1}{N} - 3 \right) \right. \right. \\ &\left. \left. + \frac{(3N^2+2N-2)}{N} + \frac{(N^2-4N-2)}{N} \ln \frac{m^2}{\mu^2} - \frac{1}{N} \ln^2 \frac{m^2}{\mu^2} \right) \frac{g^4}{16\pi^2} + \dots \right]. \end{aligned} \quad (7)$$

We define  $X_0$  and  $Y_0$  as the coefficients which do not multiply a logarithm

$$J(m) \equiv m(1 + g^2(m)X_0 + g^4(m)Y_0 + \dots). \quad (8)$$

The expansion (7) is highly scheme and scale dependent. This dependence is reduced drastically if we replace  $J(\mu)$  with  $\hat{J}$  which is the scheme and scale independent quantity associated with  $J$ , and then expand in powers of  $1/(\beta_0 \ln(\mu^2/\Lambda^2))$  rather than in  $g^2(\mu)$ . Starting with the expansion for  $J(m)$  in a general scheme, we found [2]

$$\begin{aligned}
\hat{J} = & m \left( \beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d \right)^{\gamma_0/2\beta_0} \times \left[ 1 + \frac{1}{\left( \beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d \right)} \left[ A_0 + \frac{\gamma_0 \beta_1}{2\beta_0^2} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) - \frac{d \gamma_0}{2\beta_0} \right] \right. \\
& + \frac{1}{\left( \beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d \right)^2} \left[ B_0 + A_0 \left( \frac{\gamma_0}{2\beta_0} - 1 \right) \frac{\beta_1}{\beta_0} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) + \frac{\beta_1^2}{\beta_0^2} \left( \left[ \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \right]^2 \frac{\gamma_0}{2\beta_0} \left( \frac{\gamma_0}{4\beta_0} - \frac{1}{2} \right) \right. \right. \\
& + \frac{\gamma_0}{2\beta_0} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \left. \left. - \frac{\gamma_0}{2\beta_0} \left( \frac{\beta_2}{\beta_0} - \frac{\beta_1^2}{\beta_0^2} \right) + d^2 \left( \frac{\gamma_0}{4\beta_0} \left( \frac{\gamma_0}{2\beta_0} - 1 \right) \right) + d \left( A_0 \left( 1 - \frac{\gamma_0}{2\beta_0} \right) - \frac{\gamma_0 \beta_1}{2\beta_0^2} \right. \right. \right. \\
& \left. \left. \left. + \frac{\gamma_0 \beta_1}{2\beta_0^2} \ln \left( \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{d}{\beta_0} \right) \left( 1 - \frac{\gamma_0}{2\beta_0} \right) \right) \right] + \mathcal{O} \left( \frac{1}{\beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d} \right)^3 \right] \quad (9)
\end{aligned}$$

with

$$\begin{aligned}
A_0 \equiv & X_0 - \frac{1}{2} \left( \frac{\gamma_1}{\beta_0} - \frac{\gamma_0 \beta_1}{\beta_0^2} \right) \\
B_0 \equiv & \frac{X_0}{2} \left( \frac{\gamma_0 \beta_1}{\beta_0^2} - \frac{\gamma_1}{\beta_0} \right) - \frac{\gamma_2}{4\beta_0} + \frac{\gamma_1 \beta_1}{4\beta_0^2} \\
& - \frac{\gamma_0}{4\beta_0} \left( \frac{\beta_1}{\beta_0} \right)^2 + \frac{\gamma_0 \beta_2}{4\beta_0^2} + \frac{\gamma_1^2}{8\beta_0^2} \\
& - \frac{\gamma_1 \gamma_0 \beta_1}{4\beta_0^3} + \frac{\gamma_0^2 \beta_1^2}{8\beta_0^4} + Y_0. \quad (10)
\end{aligned}$$

All the scheme and scale dependence now resides in  $d \equiv \beta_0 \ln((\Lambda_{\overline{MS}}^2/\Lambda^2)(\mu^2/m^2))$  and we can recover the original NATM by putting the naked source  $J_0$  equal to zero

$$J_0(m) \sim \hat{J}(m) = 0. \quad (11)$$

We find a nonperturbative mass gap which is a solution of

$$\begin{aligned}
1 + \frac{1}{\left( \beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d \right)} [\dots] + \frac{1}{\left( \beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + d \right)^2} [\dots] + \dots \\
= 0. \quad (12)
\end{aligned}$$

The total series is of course  $d$  independent but one can only calculate it up to a certain order in perturbation theory which will give us a mass gap that depends on  $d$ . One can check that the  $d$  dependence of the order  $n$  truncated series is  $\mathcal{O}(1/\beta_0 \ln(m^2/\Lambda_{\overline{MS}}^2) + d)^{n+1}$ . We will consider two possible

ways of fixing  $d$ . The first one reduces to a choice for  $\Lambda$ , that corresponds to a physical scheme. The second one fixes  $d$  by the principle of minimal sensitivity. In [2] we used the value of the expansion parameter  $1/(\beta_0 \ln(m^2/\Lambda_{\overline{MS}}^2) + d)$  as a source of error estimation. This works if the coefficients are of order one. Assuming that the series is asymptotic, a rather large value of the expansion parameter can still give reasonable results, as long as the complete terms in the series are small. In the next section we will show that this is indeed the case. For the 2-loop results it is better to estimate the error from the second order term than from the expansion parameter.

### III. NUMERICAL RESULTS

The exact result for the mass gap was obtained in [4]:

$$m = \frac{e^{1/2N}}{\Gamma\left(1 - \frac{1}{N}\right)} \Lambda_{PV}, \quad (13)$$

where  $\Lambda_{PV}$  is defined as the scale parameter for the running coupling, with a condition on the normalized four point function, calculated with a Pauli-Villars regularization. To obtain  $m/\Lambda_{\overline{MS}}$  we need to determine the relationship between the renormalized coupling  $g$  of the dimensional regularization  $\overline{MS}$  scheme and the coupling  $g_{PV}$  used in the Pauli-Villars scheme. This can be achieved by comparing the normalized fermion four-point function to one loop order in both schemes. We find

$$g^2 = g_{PV}^2 \left[ 1 + \frac{g_{PV}^2}{4\pi} \left( \frac{N}{2} + 1 \right) + \dots \right] \quad (14)$$

and hence

$$\Lambda_{PV} = \Lambda_{\overline{MS}} e^{-(1 + N/2)/2N}. \quad (15)$$

TABLE I. Physical scheme results.

$N$	$m_1$	$m_2$	$y/\pi$	$\Pi$	$N=\infty$	$1/N$
2	/	46.8%	3.2	-0.37	77.2%	26.1%
3	/	10.3%	2.3	-0.02	35.4%	9.4%
4	15.1%	5.3%	1.4	0.10	22.5%	4.9%
5	12.6%	5.4%	1.0	0.09	16.4%	3.0%
6	10.6%	5.5%	0.8	0.07	12.9%	2.0%
7	9.0%	5.3%	0.6	0.05	10.6%	1.5%
8	7.8%	5.0%	0.5	0.04	9.0%	1.1%
9	6.9%	4.7%	0.5	0.04	7.8%	0.9%
10	6.2%	4.4%	0.4	0.03	6.9%	0.7%
$\infty$	0%	0%	0	0	0%	0%

(See, for example, [11].) So we finally arrive at

$$m = \frac{e^{-1/4}}{\Gamma\left(1 - \frac{1}{N}\right)} \Lambda_{\overline{MS}} = e^{-1/4} \Lambda_{\overline{MS}} \left[ 1 - \frac{\gamma}{N} + \mathcal{O}\left(\frac{1}{N}\right)^2 \right] \quad (16)$$

where  $\gamma$  is the Euler-Mascheroni constant.

### A. Physical scheme

As in [2], we will define a *physical* RS on the normalized 4-point function of  $\mathcal{L}_J$  to obtain a physical value  $d_f$  for  $d$ . Demanding that  $g_f^2$  coincides with the 4-point function at zero external momentum [see Eq. (5.5) of [7]], one arrives at

$$g^2 = g_f^2 \left( 1 + \frac{3N}{8\pi} g_f^2 + \dots \right). \quad (17)$$

Taking  $\mu^2 = m^2$  leads to  $d_f = 3N/8\pi$ .<sup>1</sup> We now find the one and two loop mass gaps  $m_{f1}$  and  $m_{f2}$  as the solutions of the one and two loop truncation of Eq. (12) with  $d = d_f$ . The deviations from the exact result for  $m_{f1}$  and  $m_{f2}$  for the  $N \rightarrow \infty$  limit and the first order result in  $1/N$  have been displayed in Table I in terms of a percentage. We also provided the value of the two loop expansion parameter  $1/(2\beta_0 \ln(m_{f2}/\Lambda_{\overline{MS}}) + d_f) \equiv y$  and the second order term  $\Pi$  of the series in Eq. (12). For  $N=2,3$  we find no one-loop mass gap. All the other results lie somewhere between the  $N \rightarrow \infty$  and  $1/N$  approximations. We also observe convergence. In other words, the comparison with the exact result improves for the two loop truncation. From the  $y/\pi$  and the  $\Pi$  columns we learn that  $\Pi$  clearly gives a better indication on the size of the error.

### B. Minimal sensitivity

The equation for  $m(d)$  (12) can only be solved numerically. If we consider instead the expansion parameter  $y$  as the

free parameter one can solve it analytically to find  $m(y)$ . Indeed, we can rewrite Eq. (12) as

$$1 + y \left( A_0 - \frac{\gamma_0}{2\beta_0} k(m, y) \right) + y^2 \left( B_0 - \frac{\gamma_0}{2\beta_0} \left[ \frac{\beta_2}{\beta_0} - \left( \frac{\beta_1}{\beta_0} \right)^2 \right] \right. \\ \left. + k(m, y) \left[ A_0 \left( 1 - \frac{\gamma_0}{2\beta_0} \right) - \frac{\gamma_0 \beta_1}{2\beta_0^2} \right] \right. \\ \left. + k(m, y)^2 \left[ \frac{\gamma_0}{4\beta_0} \left( \frac{\gamma_0}{2\beta_0} - 1 \right) \right] \right) + \dots = 0 \quad (18)$$

with

$$k(m, y) \equiv d(m, y) + \frac{\beta_1}{\beta_0} \ln(\beta_0 y) \\ = \frac{1}{y} - \beta_0 \ln \frac{m^2}{\Lambda_{\overline{MS}}^2} + \frac{\beta_1}{\beta_0} \ln(\beta_0 y). \quad (19)$$

The one- and two-loop truncation of Eq. (18) is now solved easily. At one loop it is a linear equation in  $k$  and one finds  $k = (2\beta_0/\gamma_0)(1/y + A_0)$ . After substituting this into Eq. (19) we find the one loop mass gap to be

$$m_1(y) = \Lambda_{\overline{MS}}(\beta_0 y)^{\beta_1/2\beta_0^2} \exp \left[ \frac{1}{y} \left( \frac{1}{2\beta_0} - \frac{1}{\gamma_0} \right) - \frac{A_0}{\gamma_0} \right]. \quad (20)$$

The 2-loop truncation gives a quadratic equation in  $k$ , with two roots  $k_1(y), k_2(y)$ . Hence, the two solutions for the mass gap are

$$m_{2i}(y) = \Lambda_{\overline{MS}}(\beta_0 y)^{\beta_1/2\beta_0^2} \exp \left[ \frac{1}{2\beta_0} \left( \frac{1}{y} - k_i(y) \right) \right]. \quad (21)$$

The behavior of  $m_1(y)$  is more or less the same for all values of  $N$ . One observes a sharp maximum, followed by an asymptotic descent to zero. There is no region of minimal sensitivity. For  $N > 2$  the situation changes at two loops. One of the two solutions  $m_{2i}(y)$  has, in addition to the sharp maximum, a rather flat minimum. This is the point of minimal sensitivity. In Fig. 1 we plot the one- and two-loop solutions for the generic  $N=5$  case.

The other 2-loop solution is not physical since it varies enormously in the region of interest, defined as the region with acceptable estimated error, and no minimal sensitivity is found. For  $N=2$  the two-loop solution has no minimum with instead only a rather sharp maximum at 68% deviation. No true minimal sensitivity point can be identified. The results for  $N > 2$  are displayed in Table II. They are slightly better than the two-loop physical scheme. Again we find  $\Pi$  to provide a better indication on the error than  $y/\pi$ . We finally remark that also the minimal sensitivity condition can be solved exactly, to give an analytic form of the 2-loop mass gap. We will not present it here, however, since it is a large expression and does not give any new insights.

<sup>1</sup>As in [2], one can show that every physical value for  $d$  ( $\sim N$ ) gives the correct  $N \rightarrow \infty$  limit.

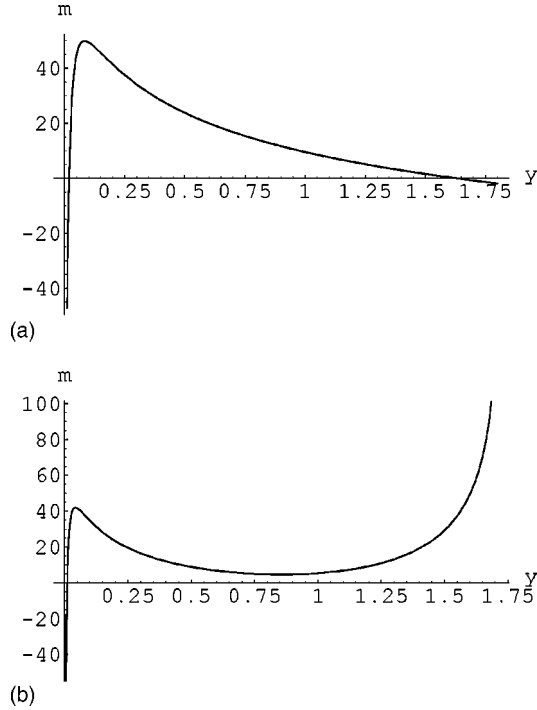


FIG. 1. (a)  $N=5, y/\pi \rightarrow ([m_1(d) - m_{\text{exact}}]/m_{\text{exact}})100$ ; (b)  $N=5, y/\pi \rightarrow ([m_2(d) - m_{\text{exact}}]/m_{\text{exact}})100$ .

#### IV. CONCLUSIONS

We have successfully applied the source inversion method to the chiral Gross-Neveu model. This required a two-loop calculation of the mass gap in the massive NATM which we carried out exactly. Comparison with the exact result for the nonperturbative mass gap gives a satisfying match. For the physical scheme, there is convergence of the 2-loop result versus the 1-loop result. The 2-loop results are good for  $N > 2$  with a 10% deviation for  $N=3$  and  $\leq 5\%$  for  $N > 3$ . The minimal sensitivity condition gives a slight improvement. As in the case of the ordinary Gross-Neveu model, the  $N=2$  result is poor. The two-loop physical scheme gives a 46% deviation. The success or failure of the method for  $N > 2/N=2$  is fairly consistent with the error estimation one obtains from the second order term in the mass gap equation. Finally, it would be worthwhile to apply the technique discussed here to other models where exact mass gaps are also available. This would have the long term aim of applying the procedure to theories where the only information on the dynamical gen-

TABLE II. Minimal sensitivity results.

$N$	$m_2$	$y/\pi$	$\Pi$
3	-10.3%	2.0	0.63
4	2.4%	1.2	0.25
5	4.7%	0.9	0.15
6	5.2%	0.7	0.10
7	5.1%	0.6	0.07
8	4.9%	0.5	0.05
9	4.6%	0.4	0.04
10	4.3%	0.4	0.04
15	3.2%	0.3	0.02
20	2.5%	0.2	0.01
$\infty$	0%	0	0

erated mass comes from say Schwinger-Dyson or lattice methods in order to ascertain how competitive the results would be.

#### ACKNOWLEDGMENTS

The two loop calculations were performed with the use of FORM [13].

#### APPENDIX: COMPUTATION OF TWO LOOP INTEGRALS

In this appendix we discuss the evaluation of the basic Feynman integrals which underlie the *exact* value of our mass gap at two loops. At one loop there is only one basic integral which is defined by

$$I = i \int_k \frac{1}{[k^2 - m^2]} \quad (\text{A1})$$

in Minkowski space where  $\int_k = \int d^\omega k / (2\pi)^2$  and it has the exact value in  $\omega$  dimensions

$$I = \frac{\Gamma(1 - \omega/2) m^{(\omega-2)/2}}{(4\pi)^{\omega/2}}. \quad (\text{A2})$$

Therefore, if  $\omega = 2 - \epsilon$  then  $I$  has a simple pole in  $\epsilon$  which is the foundation of the one loop renormalization. At two loops all contributions to the 2-point function can be reduced to several basic Feynman integrals. These are  $I^2$ ,  $\Delta(p^2)$  and  $\Delta_{\mu\nu}(p^2)$  where

$$\Delta(p^2) = i^2 \int_k \int_l \frac{1}{[(k-p)^2 - m^2][l^2 - m^2][(k-l)^2 - m^2]} \quad (\text{A3})$$

and

$$\Delta_{\mu\nu}(p^2) = i^2 \int_k \int_l \frac{k_\mu k_\nu}{[(k-p)^2 - m^2][l^2 - m^2][(k-l)^2 - m^2]} \quad (\text{A4})$$

and these latter functions only occur in the sunset topology. Other integrals with an obvious definition such as  $\Delta_\mu(p)$  and  $\Delta_{\mu\nu\sigma}(p)$  also arise but the relevant 2-point function contributions can be related to Eqs. (A3) and (A4) [12]. For instance,

$$\Delta_\mu(p) = \frac{2}{3} p_\mu \Delta(p^2), \quad p^\nu \Delta_{\mu\nu}^\mu(p) = 2p^\mu p^\nu \Delta_{\mu\nu}(p^2) - \frac{2}{3} p^2(p^2 - m^2) \Delta(p^2). \quad (\text{A5})$$

It is elementary to observe that  $\Delta(p^2)$  is finite in two dimensions. Hence, for the mass gap we only need to evaluate it in two dimensions when  $p^2 = m^2$ . To do this we follow the Feynman parameter approach of [14] which gives

$$\Delta(p^2) = \frac{\Gamma(3-\omega)}{(4\pi)^\omega} \int_0^1 dx \int_0^1 dy \frac{[xy(1-y)]^{1-\omega/2}}{[y(1-y)(x(1-x)p^2 - (1-x)m^2) - xm^2]^{3-\omega}} \quad (\text{A6})$$

in  $\omega$  dimensions after carrying out the momentum integrations. Restricting to two dimensions the  $y$  integration can be performed from an integral representation of the hypergeometric function,  ${}_2F_1(a, b; c; z)$ , giving

$$\begin{aligned} \Delta(p^2)|_{\omega=2} &= \frac{1}{(2\pi)^2} \int_0^1 \frac{dx}{[x(1-x)p^2 - (1+3x)m^2]} \\ &\times {}_2F_1\left(1, \frac{3}{2}; \frac{3}{2}; \frac{[x(1-x)p^2 - (1-x)m^2]}{[x(1-x)p^2 - (1+3x)m^2]}\right). \end{aligned} \quad (\text{A7})$$

Next we set  $p^2 = m^2$  in the two dimensional integral to obtain

$$\Delta(p^2)|_{\omega=2, p^2=m^2} = -\frac{1}{(2\pi m)^2} \int_0^1 \frac{dx}{(1+x)^2} {}_2F_1\left(1, \frac{3}{2}; \frac{3}{2}; \frac{(1-x)^2}{(1+x)^2}\right) \quad (\text{A8})$$

which reduces to

$$\Delta(p^2)|_{\omega=2, p^2=m^2} = \frac{1}{8\pi^2 m^2} \int_0^1 dx \frac{\ln x}{(1-x^2)}. \quad (\text{A9})$$

The final integral can now be calculated exactly [15] to produce

$$\Delta(p^2)|_{\omega=2, p^2=m^2} = -\frac{3\zeta(2)}{32\pi^2 m^2}. \quad (\text{A10})$$

The remaining integral which arises in the sunset topology occurs with two Lorentz contractions. First, in  $\omega$  dimensions without setting the on-shell condition it is straightforward to show that [12]

$$\Delta_\mu^\mu(p^2) = I^2 + \frac{1}{3}(p^2 + 3m^2)\Delta(p^2). \quad (\text{A11})$$

Although the contraction of Eq. (A4) with  $p_\mu p_\nu$  is also divergent it cannot be written in a similar closed form. However, its divergent part is known to be  $p^2 I^2 / \omega$  [12]. Therefore,

$$F_\Delta(p^2) = \omega p^\mu p^\nu \Delta_{\mu\nu}(p^2) - p^2 \Delta_\mu^\mu(p^2) \quad (\text{A12})$$

will be finite in two dimensions and can be evaluated exactly when the on-shell condition is set similar to the derivation of Eq. (A10). With the same Feynman parametrization as Eq. (A6) we have

$$\begin{aligned} F_\Delta(p^2) &= \frac{(\omega-1)\Gamma(3-\omega)(p^2)^2}{(4\pi)^\omega} \\ &\times \int_0^1 dx \int_0^1 dy \frac{(1-x)^2 [xy(1-y)]^{1-\omega/2}}{[y(1-y)(x(1-x)p^2 - (1-x)m^2) - xm^2]^{3-\omega}}. \end{aligned} \quad (\text{A13})$$

Hence, using the properties of the hypergeometric function again we find

$$F_{\Delta}(p^2)|_{\omega=2, p^2=m^2} = \frac{m^2}{8\pi^2} \int_0^1 dx \frac{(1-x)}{(1+x)} \ln x \quad (\text{A14})$$

leading to [15]

$$F_{\Delta}(p^2)|_{\omega=2, p^2=m^2} = \frac{m^2}{8\pi^2} [1 - \zeta(2)]. \quad (\text{A15})$$

Hence, all integrals in the full 2-point function can be written in terms of  $I^2$ , Eqs. (A10) and (A15).

We have checked that the values obtained here for the finite parts of  $\Delta(p^2)$  and  $\Delta_{\mu\nu}(p^2)$  agree with the numerical values given in [10]. For instance, repeating the calculation which leads to Eq. (4.5) of [10] we find that  $[0.737775 - \pi^2/96]$  corresponds to  $[\zeta(2)/2 - 3/16]$ .

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